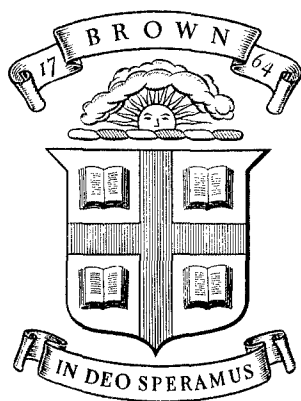


BU  
ARPA-E-37

AD 651 348



Division of Engineering  
BROWN UNIVERSITY  
PROVIDENCE, R. I.

# IMPULSIVE LOADING OF A SIMPLY SUPPORTED CIRCULAR PLATE

N. JONES

AD651 348

Department of Defense  
Advanced Research Projects Agency  
Contract SD-86  
Material Research Program

ARPA SD-86  
Report 37

February 1967

TECHNICAL LIBRARY  
BLDG 813  
ABERDEEN PROVING GROUND, MD.  
STEAP-TL

3 cop  
1-TL  
1-Mr. Zuker  
1-Dr. F. Bennett

BU  
ARPA-E-37

IMPULSIVE LOADING OF A SIMPLY SUPPORTED CIRCULAR PLATE\*

by

Norman Jones

Division of Engineering  
Brown University  
Providence, Rhode Island

February 1967

\* The research reported here was supported by the Advanced Research Project Agency, Department of Defense, under Contract Number SD-86.

20060223349

TECHNICAL LIBRARY  
BLDG. 818  
ABERDEEN PROVING GROUND, MD.  
STAP-TL

# IMPULSIVE LOADING OF A SIMPLY SUPPORTED CIRCULAR PLATE

by

Norman Jones\*

## Abstract


It is clear from a survey of literature on the dynamic deformation of rigid-plastic plates that most work has been focused on plates in which either membrane forces or bending moments alone are considered important, while the combined effect of membrane forces and bending moments on the behavior of plates under static loads is fairly well established. This article, therefore, is concerned with the behavior of circular plates loaded dynamically and with deflections in the range where both bending moments and membrane forces are important.

A general theoretical procedure is developed from the equations for large deflections of plates and a simplified yield condition due to Hodge. It can be shown that these general equations may be reduced to give the predictions of Onat and Haythornthwaite for static loading and to those of Wang for the dynamic case with bending moments only if the rigid, perfectly plastic material from which the plate is made yields according to Tresca and any strain rate effects are disregarded.

The results obtained when solving the governing equations for the particular case of a simply supported circular plate loaded with a uniform impulsive velocity are found to compare favorably with the corresponding experimental values recorded by Florence. Although this study is of general interest, it is thought in particular that it should assist in the interpretation of the dynamic biaxial stress-strain characteristics of materials recorded on diaphragms fitted in impact tubes.

---

\* Assistant Professor of Engineering (Research), Brown University



## Notation

$A_s, B_s, C_s$	defined by equations (52), (53) and (55)
$E_s, F_s$	arbitrary constants of integration
$G_s, H_s$	defined by equations (45) and (46)
$H$	plate thickness
$I$	impulse per unit area of plate
$J_n( )$	Bessel function of order $n$
$K_s$	defined by equation (54)
$M_o$	$\sigma_o H^2/4$
$M_r, M_\theta$	radial and circumferential bending moments per unit length
$N_o$	$\sigma_o H$
$N_r, N_\theta$	radial and circumferential membrane forces per unit length
$Q$	transverse shear force per unit length of plate
$R$	outside radius of plate
$R_r, R_\theta$	principal radii of curvature
$V_o$	initial velocity of plate
$Y_o( )$	Neumann's Bessel function of the second kind of zero order
$k$	uniform distributed pressure per unit area of undeformed plate
$m_r, m_\theta$	dimensionless bending moments $M_r/M_o, M_\theta/M_o$
$n_r, n_\theta$	dimensionless membrane forces $N_r/N_o, N_\theta/N_o$
$p$	$-k \sin \phi$

# Notation (continued)

$q$	$-k \cos \phi$
$r$	radial coordinate of plate
$t$	time
$t_1$	$\frac{\mu V_o R^2}{12 M_o}$
$\bar{t}$	duration of first stage of deformation
$t_s$	duration of entire deformation corresponding to the greatest value from equation (48)
$u$	displacement in direction $r$ of undeformed plate
$w$	transverse deflection perpendicular to undeformed plate
$w_f$	final deformation
$x$	$r/R$
$y$	$\alpha_s r/R$
$\alpha_r$	$1 + \epsilon_r$
$\alpha_s$	$s$ roots of equation $J_o(\alpha_s) = 0$
$\alpha_\theta$	$u + r$
$\beta$	$\mu V_o^2 R / M_o$
$\gamma_s$	$\lambda_s (N_o / \mu)^{1/2}$
$\epsilon$	$H/R$
$\epsilon_r, \epsilon_\theta$	radial and circumferential strains
$\theta$	circumferential coordinate lying in plate
$\lambda_s$	$\alpha_s / R$
$\mu$	mass per unit area of plate

Notation (continued)

$\rho$	radius of hinge
$\sigma_o$	yield stress in simple tension
$\phi$	slope of the mid-plane of a plate measured in a plane which passes through $r = 0$ and is perpendicular to the plate surface
$\Delta$	$v_o \bar{t}/R$
$(\dot{\phantom{a}})$	$\frac{\partial}{\partial t} ( )$
$(\phantom{a})'$	$\frac{\partial}{\partial r} ( )$

## Introduction

During the second world war Taylor (1), Hudson (2), Richardson and Kirkwood (3) and others (4) were the first to conduct experimental and theoretical studies into the influence of dynamic loads on the behavior of thin disks or circular plates. Further work on membranes has been reported by Frederick (5), who proposed a mechanism of behavior somewhat similar to that of Hudson (2), and Griffith and Vanzant (6), who recorded dynamic load-carrying capacities significantly greater than the corresponding static values. It is also evident from their results that, at high loading rates, elements of such plates tend to move in a transverse sense which ensues in smaller circumferential strains than those expected from static tests.

Hopkins and Prager (7) contributed some general information on the dynamic behavior of plates and studied the particular problem of a simply supported circular plate subjected to a uniformly distributed load which is brought on suddenly, and, after being kept constant during a certain interval of time, is removed suddenly. The plate is made of a rigid, perfectly plastic material, which is assumed to obey the Tresca yield condition and associated flow rule. Any membrane forces which may arise during deformation are disregarded. Making the same assumptions and using a similar but more complicated mechanism of behavior, Florence (8) solved the problem of a circular plate clamped around its outer edge and loaded with a central rectangular pulse. Wang and Hopkins (9) studied the behavior of a circular plate with a transverse velocity imparted instantaneously to the entire plate except at the built-in outer edge, where the velocity is zero. Wang (10) found that this analysis simplifies considerably when the plate is simply supported around its outer edge. Florence (11) subjected some simply supported circular plates to uniformly distributed

impulses and found, particularly for large impulses, that the theoretical analysis in (10) overestimated considerably the final deformed shapes. This discrepancy, however, arises mainly because the nature of the assumptions involved in the development of the analysis in (10) limits application of the results to plates with small final deformations.

Perzyna (12) examined the influence of pulses of arbitrary shape by developing further the theory of Hopkins and Prager (7) to show that, for a given impulse, the character of the pressure-time function has little influence on the final shape of the plate. In passing, however, it is worth noting that Sankaranarayanan (13) showed that the pulse shape influenced considerably the final deformation of plastic spherical caps subjected to impact pressure loads, while Symonds (14) found somewhat less sensitivity to the pulse shape in beams.

Hopkins (15) developed a more general theory for plates loaded transversely with non-symmetrical loads, but disregarded any membrane forces and solved no particular problems.

Shapiro (16), who was the first to examine the dynamic behavior of annular plates, studied the problem of a circular plate supported rigidly around an inner radius and loaded with a circular ring of impulsive velocity around its outer edge. Recently, Florence (17) reviewed this problem in order to assess the relative contributions of membrane forces and bending moments to the formation of the final deformed shape. It was found that a solution which considered interaction between the circumferential membrane force and circumferential bending moment was closer to the experimental values than the solutions for bending moment only and membrane force only, both of which overestimated considerably the final deformations. Witmer, Balmer, Leech and Pian (18) developed a numerical method and a computer program, the predictions of which compare favorably with experimental



values recorded for the large dynamic deformations of beams, rings, plates and shells.

Boyd (19) reconsidered the problem of dynamic deformation of a circular membrane and solved the governing equations numerically for a general form of symmetrical pressure loading which permits variations across the plate and with time. Although any contributions arising from bending moments were neglected, this method predicted results similar to the corresponding values computed in Ref. (18). Boyd (19) and Frederick (5) investigated the deformations of membranes made from a strain-hardening material and discovered that a simplified rigid, perfectly plastic analysis provides a remarkably accurate approximation to the true behavior. Munday and Newitt (20) examined carefully the dynamic behavior of clamped copper membranes, and presented some photographs taken from a high-speed cine film recorded during the passage of a hinge which formed around the outer edge of a plate at the first instant of impact and travelled inwards towards the center, where it remained until the plate came to rest. For an impulsive loading which imparts to the plate a velocity increasing from zero at the first instant of impact to a maximum value at the center some time later, Munday and Newitt observed that the final deformed shapes are almost identical for all plates irrespective of thickness, diameter and load. Johnson, Poynton, Singh and Travis (21) performed some underwater explosive stretch-forming experiments on clamped circular blanks, and measured the thickness strains across the plates and final deformed shapes for various charge strengths.

— It is of passing interest to note that the method of Martin and Symonds (22) can be used to predict the maximum deflections and time bounds for rigid-plastic plates loaded impulsively. Although the time bounds compare rather

well with those predicted in (9, 10), the deflections are overestimated by one third for the simply supported case and somewhat less for the plate with clamped edges. It is worth emphasizing, however, that these results are obtained from a few lines of arithmetic whereas the solution in (9) in particular is time consuming.

Cooper and Shifrin (23) and Haythornthwaite and Onat (24) measured the static load-carrying capacity of initially flat circular plates, and observed that the bending only solution of Hopkins and Prager (25) underestimates considerably the load which could be supported if deflections of the order of the plate thickness or larger are permitted. In order to explain the strengthening effect under static loads, Onat and Haythornthwaite (26) considered both bending moments and membrane forces in an analysis which reduces to the bending only theory of Ref. (25) at very small deflections and to the behavior as a membrane at large deflections. Symonds and Mentel (27) have shown that a simple analysis of plastic beams constrained axially and loaded with a transverse pressure impulse becomes unrealistic if deflections of the order of the beam thickness are permitted and the influence of axial forces is disregarded. The authors describe the transition of clamped and simply supported beams from an initial simple behavior to a final stage in which deformations are governed primarily by catenary effects.

It is clear from the foregoing survey of literature that most effort has been concentrated on the dynamic deformation of plates in which either membrane forces (1-5, 19) or bending moments (7-10, 12, 15, 16) alone are believed to be important. Moreover, with the exception of the computing work in (18) and the analysis by Florence (17) for an annular plate, no investigations have been conducted into the interaction effects between membrane forces

and bending moments, while for static loading the behavior during all stages is fairly well understood (26).

The object of this present article, therefore, is an attempt to link the two distinct stages of plastic strain and describe the behavior of plates dynamically loaded with deflections in the range where both bending moments and membrane forces are important. Subsequently, it is found that the theoretical analysis presented herein predicts with reasonable accuracy the final deformations recorded by Florence (11) on a simply supported circular plate subjected to a uniform impulse.

### Equilibrium Equations

Introducing the inertia terms into the equilibrium equations developed by Reissner (28) for the finite deflection of a circular plate loaded statically gives

$$(\alpha_\theta N_r)' - \alpha_\theta' N_\theta - \alpha_r \alpha_\theta Q/R_r + \alpha_r \alpha_\theta p - \mu \alpha_\theta \ddot{w}' - \mu \alpha_\theta \alpha_\theta' \ddot{u} = 0 \quad (1)$$

$$(\alpha_\theta Q)' + \alpha_r \alpha_\theta [N_r/R_r + N_\theta/R_\theta] + \alpha_r \alpha_\theta q + \mu \alpha_\theta \alpha_\theta' \ddot{w} - \mu \alpha_\theta \ddot{u} w' = 0 \quad (2)$$

$$(\alpha_\theta M_r)' - \alpha_\theta' M_\theta - \alpha_r \alpha_\theta Q = 0 \quad (3)$$

provided the rotary inertia effect is disregarded, and

$$\alpha_r = 1 + \epsilon_r,$$

$$\alpha_\theta = r + u = r(1 + \epsilon_\theta),$$

$$1/R_r = \phi'/(1 + \epsilon_r),$$

$$1/R_\theta = \sin\phi/r ,$$

$$(\dot{\phantom{x}}) = \frac{\partial}{\partial t} (\phantom{x}) , \quad t = \text{time}$$

$$(\phantom{x})' = \frac{\partial}{\partial r} (\phantom{x}) ,$$

The positive directions of the various quantities are indicated in Fig. 1.

Since we are interested in the interaction between membrane forces and bending moments, then we may follow Reissner (28) and limit equations (1) - (3) to plates having small strains and deflections which are not too large. Thus, using

$$\alpha_\theta = r, \alpha_r = 1, 1/R_r = \phi', 1/R_\theta = \sin\phi/r, \text{ and } \alpha_\theta' = \cos\phi$$

equations (1) - (3) become

$$(rN_r)' - \cos\phi N_\theta - r\phi'Q + rp - \mu r\ddot{w}w' - \mu r \cos\phi\ddot{u} = 0 \quad (4)$$

$$(rQ)' + rN_r\phi' + \sin\phi N_\theta + rq + \mu r \cos\phi\ddot{w} - \mu r\ddot{u}w' = 0 \quad (5)$$

$$(rM_r)' - \cos\phi M_\theta - rQ = 0 \quad (6)$$

If it is further assumed that  $\cos\phi = 1$  and  $\sin\phi = -w'$ , and recognized that  $r\phi'Q$  and  $rN_r\phi'$  are small in comparison with the remaining terms, then equations (4) - (6) can be rewritten in the form

$$rn_r' + n_r - n_\theta = -rkw'/N_0 + \mu r\ddot{w}w'/N_0 + \mu r\ddot{u}/N_0 \quad (7)$$

$$rm_r'' + 2m_r' - m_\theta' - 4n_\theta w'/H = rk/M_0 - \mu r\ddot{w}/M_0 + \mu r\ddot{u}w'/M_0 \quad (8)$$

where

$$n_{r,\theta} = N_{r,\theta}/N_0 ,$$

$$m_{r,\theta} = M_{r,\theta}/M_0 ,$$

and the second order terms  $\phi'w'$  are disregarded. In view of the observations of Griffith and Vanzant (6) that the material of a circular plate tends to move in a transverse sense at high rates of dynamic loading, then we assume hereafter that

$$\ddot{u} = 0 \quad (9)$$

### Yield Condition

It is necessary to employ a four-dimensional yield surface between the generalized stresses  $n_r, n_\theta, m_r$  and  $m_\theta$ , in order to solve the equilibrium equations (7) and (8) for a plate made from a rigid-plastic material. Onat and Prager (29) have derived equations for the load-carrying capacity of shells of revolution made from a rigid, perfectly plastic material which obeys Tresca's yield condition and the associated flow rule. However, these relations are very difficult to handle for other than simple problems. Hodge (30), therefore, proposed a two-moment limited interaction surface which maintains all interaction between force and force and moment and moment, but neglects all interactions between force and moment. If it is assumed that yielding is controlled by the Tresca criteria, the result is a linear surface in four-dimensional space; and the equations of the twelve planes are indicated in Fig. 2.

The yield condition proposed by Hodge (30) was used to solve the equilibrium equations (7) and (8) for a simply supported plate loaded with a static pressure distributed uniformly over a central circular area. This method predicted a variation of load carrying capacity very similar to that forecasted by Onat and Haythorthwaite (26) for deflections greater than  $H/2$ .

### Mechanism of Deformation

The notion of a travelling hinge has been used successfully for establishing the dynamic behavior of beams and cantilevers (27, 31, etc.). Furthermore, Munday and Newitt (20) observed that at the first instant of impact a hinge develops at the supports of a circular plate and travels inward towards the center, where it remains until the plates come to rest. It seems reasonable, therefore, to employ herein the similar mechanism of behavior proposed by Wang (10) for a simply supported circular plate loaded impulsively.

Thus at  $t \leq 0$ , the plate travels with an initial uniform velocity  $v_0$ , and a hinge is assumed to form at the support radius  $r = R$  at the instant  $t = 0$ . Some time later ( $t$ ), the hinge will reach a radius  $\rho(t)$ , where  $0 < \rho(t) < R$ , and divide the plate into two distinct zones as indicated in Fig. 3 where

$$\dot{w} = V_0, \quad \text{for } 0 \leq r \leq \rho(t) \quad (10)$$

and

$$\dot{w} = V_0(R - r)/(R - \rho), \quad \text{for } \rho(t) \leq r \leq R \quad (11)$$

It is assumed that the kinetic energy of the plate for the moment when the travelling hinge reaches radius  $\rho = 0$  is dissipated during a second stage of deformation in which the hinge remains stationary at  $r = 0$ .

### Expression for Slope ( $w'$ )

If  $\rho(t) \leq r \leq R$ , then using (10) and (11) the transverse deflection of the plate at time  $t$  is

$$w = V_0 t(r) + \int_{t(r)}^t \frac{\partial w}{\partial t} dt \quad (12)$$

where  $V_0$  is the initial uniform velocity of the plate at  $t = 0$ ;  $t(r)$  is the time taken for a travelling hinge to reach radius  $r$ , provided  $\rho(t) \leq r \leq R$ , while  $t$  is the time at which the hinge reaches radius  $\rho$ .

Letting

$$\frac{\partial w}{\partial t} = \bar{R} T$$

where  $\bar{R}$  is a function of  $r$  alone, and  $T$  is a function of  $t$  alone permits (12) to be rewritten

$$w = V_0 t(r) + \bar{R} \int_{t(r)}^t T dt$$

whence,

$$\frac{\partial w}{\partial r} = V_0 \frac{\partial}{\partial r} \{t(r)\} + \frac{\partial}{\partial r} (\bar{R}) \int_{t(r)}^t T dt + \bar{R} \frac{\partial}{\partial r} \left\{ \int_{t(r)}^t T dt \right\} \quad (13)$$

### Analysis of a Simply Supported Circular Plate Loaded Impulsively

#### (i) First Stage of Deformation

It can be shown that the yield conditions,

$$m_r = m_\theta = -1, \quad n_r = n_\theta = 1, \quad \text{for } 0 \leq r \leq \rho \quad (14)$$

and

$$m_\theta = -1, \quad 0 \leq m_r \leq -1, \quad n_r = 1, \quad 0 < n_\theta \leq 1, \quad \text{for } \rho \leq r \leq R \quad (15)$$

indicated in Fig. 2, are consistent with equations (10) and (11). Equations (10) and (14) satisfy exactly equations (7), (8), (9) and (13), while (7), (8), (9) and (15) give

$$\frac{\partial}{\partial r} (r^2 m_r') = \frac{4r}{H} \frac{\partial w}{\partial r} - \frac{\mu}{M_0} \frac{r^2}{\partial t^2} \frac{\partial^2 w}{\partial t^2} + \frac{kr^2}{M_0} \quad (16)$$

provided

$$\rho(t) \leq r \leq R, \quad \text{and} \quad (w')^2 \ll 1.$$

If  $\partial w / \partial r = 0$  and  $k = 0$ , then for the first stage of deformation it can be shown that equations (11) and (16) yield the following time function,

$$t = t_1 \left[ 1 - \frac{\rho}{R} - \left( \frac{\rho}{R} \right)^2 + \left( \frac{\rho}{R} \right)^3 \right] \quad (17)$$

which is identical to that obtained by Wang (10) who neglected the effect of membrane forces.

Since one would expect the influence of the bending moments to dominate over the action of the membrane forces during the first stage, then the form of the time function for  $\partial w / \partial r \neq 0$  will be taken as

$$t = \bar{t} \left[ 1 - \frac{\rho}{R} - \left( \frac{\rho}{R} \right)^2 + \left( \frac{\rho}{R} \right)^3 \right] \quad (18)$$

where  $\bar{t}$  is a constant to be determined later. Differentiating equation (18) with respect to time gives

$$1/(R - \rho) = -\dot{\rho} \bar{t} (3\rho + R)/R^3 \quad (19)$$

which allows (13) to be rewritten in the form

$$\frac{\partial w}{\partial r} = \frac{V_0 \bar{t}}{R^3} \left( \frac{3\rho^2}{2} + R\rho - \frac{3r^2}{2} - Rr \right), \quad \text{for} \quad \rho(t) \leq r \leq R. \quad (20)$$



Substituting equations (11) and (20) into (16) yields

$$\frac{\partial}{\partial r} (r^2 m_r') = \frac{4V_o \bar{t}}{R^3 H} \left( \frac{3}{2} \rho^2 r + R \rho r - \frac{3}{2} r^3 - R r^2 \right) - \frac{\mu \dot{\rho} V_o r^2 (R - r)}{M_o (R - \rho)^2} \quad (21)$$

which when integrated with the condition that  $m_r = -1$  at  $r = \rho$  gives

$$m_r = \frac{4V_o \bar{t}}{R^3 H} \left( \frac{3}{4} \rho^2 r + \frac{R}{2} \rho r - \frac{r^3}{8} - \frac{R r^2}{6} + \frac{3\rho^4}{8r} + \frac{R \rho^3}{6r} - \rho^3 - \frac{R \rho^2}{2} \right) - \frac{\mu \dot{\rho} V_o}{M_o (R - \rho)^2} \left( \frac{R}{6} r^2 - \frac{r^3}{12} + \frac{R \rho^3}{3r} - \frac{\rho^4}{4r} - \frac{R \rho^2}{2} + \frac{\rho^3}{3} \right) - 1 \quad (22)$$

Since  $m_r = 0$  at  $r = R$ , equation (22) yields

$$\frac{7V_o \bar{t}}{6H} - \frac{\mu V_o R^2}{12M_o \bar{t}} + 1 = 0, \quad \text{at } \rho = 0 \quad (23)$$

the solution of which can be written as

$$V_o \bar{t} = R \left\{ -\frac{3H}{7R} + \frac{1}{2} \left( \frac{2\mu H V_o^2}{7M_o} \right)^{1/2} \left( 1 + \frac{18H M_o}{7\mu V_o^2 R^2} \right)^{1/2} \right\} \quad (24)$$

or,

$$\Delta = -\frac{3}{7} \epsilon + \left( \frac{\beta \epsilon}{14} \right)^{1/2} \left( 1 + \frac{9}{7} \frac{\epsilon}{\beta} + \dots \right) \quad (25)$$

where,

$$\epsilon = H/R \quad (26)$$

$$\beta = \mu V_o^2 R / M_o \quad (27)$$

and

$$\Delta = V_o \bar{t} / R \quad (28)$$

$V_0 \bar{t}$  is the deflection of the center of the plate at the conclusion of the first stage ( $\rho = 0$ ).

It can be shown that the deflected form of the plate at  $t = \bar{t}$  is

$$w = V_0 \bar{t} \left[ 1 - \frac{1}{2} \left( \frac{r}{R} \right)^2 - \frac{1}{2} \left( \frac{r}{R} \right)^3 \right] \quad (29)$$

(ii) Second Stage of Deformation

Now, one would expect the membrane forces to dominate over the bending moments during the second stage of deformation throughout which the hinge remains stationary at  $r = 0$ . Thus, assuming  $M_r = M_\theta = Q = 0$  and considering the case when  $k = 0$ , equations (1), (2) and (9) become

$$(r n_r)' - \cos \phi n_\theta = \mu r \ddot{w} w' / N_0 \quad (30)$$

and

$$r \phi' n_r + \sin \phi n_\theta = -\mu r \cos \phi \ddot{w} / N_0. \quad (31)$$

Utilizing the same yield condition as used previously for  $\rho \leq r \leq R$  (15) permits equations (30) and (31) to be rewritten as one equation,

$$w'' + w'/r = \mu \ddot{w} / N_0 \quad (32)$$

since,

$$(r n_r)' = r n_r' + \cos \phi n_r$$

$$\cos \phi = 1, \quad \sin \phi = -w', \quad \text{and} \quad (w')^2 \ll 1.$$

The general solution of (32) may be written in the following form after

Bowman (32)

$$w = \{A J_0(\lambda_s r) + B Y_0(\lambda_s r)\} \{C \cos(\sqrt{\frac{N_0}{\mu}} \lambda_s t) + D \sin(\sqrt{\frac{N_0}{\mu}} \lambda_s t)\} \quad (33)$$

for  $\bar{t} \leq t \leq t_s$

or

$$w = \{E_s \cos(\sqrt{\frac{N_0}{\mu}} \lambda_s t) + F_s \sin(\sqrt{\frac{N_0}{\mu}} \lambda_s t)\} J_0(\lambda_s r) \quad (34)$$

in order to maintain finite deflections at  $r = 0$ .

The deflection  $w$  must be zero at the simply supported edge for all values of  $t$ . This restriction on (34) yields the relation,

$$\lambda_s = \frac{\alpha_s}{R}, \quad s = 1, 2, \dots, \infty \quad (35)$$

where  $\alpha_s$  are the roots of the equation  $J_0(\alpha_s) = 0$ . Thus,

$$w = \sum_{s=1}^{\infty} \{E_s \cos(\gamma_s t) + F_s \sin(\gamma_s t)\} J_0(\lambda_s r) \quad (36)$$

where

$$\gamma_s = \sqrt{\frac{N_0}{\mu}} \lambda_s \quad (37)$$

The deflection  $w$  and velocity  $\dot{w}$  given by (36) and its derivative must match the values acquired by the plate at the end of the first stage given by (29) and (11) with  $\rho = 0$  and  $t = \bar{t}$ .

Hence,

$$v_0 \bar{t} [1 - \frac{x^2}{2} - \frac{x^3}{2}] = \sum_{s=1}^{\infty} \{E_s \cos(\gamma_s \bar{t}) + F_s \sin(\gamma_s \bar{t})\} J_0(\alpha_s x) \quad (38)$$

and

$$V_o(1-x) = \sum_{s=1}^{\infty} \{-\gamma_s E_s \sin(\gamma_s \bar{t}) + \gamma_s F_s \cos(\gamma_s \bar{t})\} J_o(\alpha_s x) \quad (39)$$

where,

$$x = r/R . \quad (40)$$

Multiplying both sides of (38) and (39) by  $x J_o(\alpha_s x) dx$  and integrating over the limits  $x = 1$  and  $x = 0$ , it can be shown with the aid of Lommel's integrals (Ref. 32), that

$$G_s = \frac{2}{J_1^2(\alpha_s)} \int_0^1 x V_o \bar{t} (1 - \frac{x^2}{2} - \frac{x^3}{2}) J_o(\alpha_s x) dx \quad (41)$$

and

$$H_s = \frac{2}{J_1^2(\alpha_s)} \int_0^1 x V_o (1 - x) J_o(\alpha_s x) dx \quad (42)$$

where

$$G_s = E_s \cos(\gamma_s \bar{t}) + F_s \sin(\gamma_s \bar{t}) \quad (43)$$

and

$$H_s = -\gamma_s E_s \sin(\gamma_s \bar{t}) + \gamma_s F_s \cos(\gamma_s \bar{t}) \quad (44)$$

If  $y = \alpha_s x$ , equations (41) and (42) become

$$G_s = \frac{V_o \bar{t}}{J_1^2(\alpha_s) \alpha_s^3} \{13 J_1(\alpha_s) - \frac{9}{\alpha_s^2} \int_0^{\alpha_s} J_o(y) dy\} \quad (45)$$

and

$$H_s = \frac{2V_o}{J_1^2(\alpha_s) \alpha_s^3} \int_0^{\alpha_s} J_o(y) dy \quad (46)$$

where values of the Bessel functions are tabulated in references (33) and (34). Schmidt (35) tabulates the integral

$$\int_0^{\alpha_s} J_0(y) dy \quad \text{for} \quad 10 \leq y \leq 50 .$$

The plate will come to rest when  $\dot{w} = 0$ . Hence,

$$\sum_{s=1}^{\infty} \{-\gamma_s E_s \sin(\gamma_s t_s) + \gamma_s F_s \cos(\gamma_s t_s)\} J_0(\lambda_s r) = 0 \quad (47)$$

or

$$\tan(\gamma_s t_s) = F_s / E_s . \quad (48)$$

Thus the final deformed shape  $w_f$  at  $t = t_s$  is

$$w_f = \sum_{s=1}^{\infty} \sqrt{E_s^2 + F_s^2} J_0(\lambda_s r) \quad (49)$$

or

$$w_f = \sum_{s=1}^{\infty} \sqrt{G_s^2 + (H_s/\gamma_s)^2} J_0(\lambda_s r) \quad (50)$$

which may be expressed in the form

$$w_f = R \sum_{s=1}^{\infty} A_s \sqrt{(\Delta B_s)^2 + \beta \epsilon c_s^2} J_0(\lambda_s r) \quad (51)$$

where

$$A_s = 1 / \{J_1^2(\alpha_s) \alpha_s^3\} , \quad (52)$$

$$B_s = 13 J_1(\alpha_s) - 9 K_s / \alpha_s^2 , \quad (53)$$

$$K_s = \int_0^{\alpha_s} J_o(y) dy \quad (54)$$

and

$$C_s = K_s / \alpha_s . \quad (55)$$

### Conclusions

A general theoretical procedure which retains both membrane forces and bending moments, but neglects strain rate effects, is developed herein in order to describe the behavior of rigid, perfectly plastic circular plates loaded with axisymmetrical dynamic loads. It can be shown that this analysis may be reduced to the predictions of Onat and Haythorthwaite (26) for static loading of simply supported plates and to those of Wang (10) for the dynamic case with bending moments only.

It is evident from Fig. 4 that the maximum deflection at  $r = 0$  predicted by the first five terms of equation (51) compares favorably over a wide range of uniformly distributed impulses with the experimental values observed by Florence (11) on simply supported circular aluminium plates having  $R/H = 16$ . Theoretical predictions of the maximum plate deflections are presented in dimensionless form in Fig. 5 for various plate parameters and magnitudes of impulse.

It is thought that the method developed could be used to describe the behavior of plates having other support conditions and various dynamic loads.

### Acknowledgments

The work reported herein was supported by the Advanced Research Project Agency, Department of Defense, under contract number SD-86 awarded to Brown University.

The author wishes to take this opportunity to express his appreciation to Professor P. S. Symonds for frequent discussions throughout the execution of this work.

Thanks are also due to Miss E. Cerutti for computing the final results and to the National Science Foundation (Grant number GP-4825) for making funds available to cover the costs of machine time.

### References

1. Taylor, G. I., "The Distortion Under Pressure of a Diaphragm which is Clamped Along its Edge and Stressed Beyond the Elastic Limit," Underwater Explosion Research, Vol. III, The Damage Process, pp. 107-121, Office of Naval Research, Dept. of the Navy, 1950 (originally written in Sept. 1942).
2. Hudson, G. E., "A Theory of the Dynamic Plastic Deformation of a Thin Diaphragm," Jnl. Applied Phys., Vol. 22, No. 1, Jan. 1951.
3. Richardson, J. M., and Kirkwood, J. G., "Theory of the Plastic Deformation of Thin Plates by Underwater Explosions," Underwater Explosion Research, O.N.R., Vol. III, The Damage Process, 1950, pp. 305-421.
4. Underwater Explosion Research, A Compendium of British and American Reports, O.N.R., Vol. 3, The Damage Process, 1950.
5. Frederick, D., "A Simplified Analysis of Circular Membranes Subjected to an Impulsive Loading Producing Large Plastic Deformations," Proc. 4th Annual Conf. on Solid Mechanics, University of Texas, Austin, Texas, Sept. 1959.
6. Griffith, J., and Vanzant, H., "Large Deformation of Circular Membranes Under Static and Dynamic Loading," Paper No. 702, 1st Internat. Congress on Exptl. Mech., New York, Nov. 1-3, 1961.

7. Hopkins, H. G., and Prager, W., "On the Dynamics of Plastic Circular Plates," ZAMP (Journal of Applied Mathematics and Physics), Vol. 5, No. 4, pp. 317-330.
8. Florence, A. L., "Clamped Circular Rigid-Plastic Plates Under Central Blast Loading," Int. J. Solids & Structures, Vol. 2, pp. 319-335, 1966.
9. Wang, A. J., and Hopkins, H. G., "On the Plastic Deformation of Built-in Circular Plates Under Impulsive Load," Jnl. Mech. & Physics of Solids, Vol. 3, No. 1, pp. 22-37, 1954.
10. Wang, A. J., "The Permanent Deflection of a Plastic Plate Under Blast Loading," Jnl. App. Mechanics, Vol. 22, 1955, pp. 375-376.
11. Florence, A. L., "Circular Plate Under a Uniformly Distributed Impulse," Int. J. Solids Structures, Vol. 2, pp. 37-47, 1966.
12. Perzyna, P., "Dynamic Load Carrying Capacity of a Circular Plate," Arch. Mach. Stos., Vol. 10, No. 5, pp. 635-647, 1958.
13. Sankaranarayanan, R., "On the Impact Pressure Loading of a Plastic Spherical Cap," Journal of Applied Mechanics, Vol. 33, No. 3, Sept. 1966, pp. 704-706.
14. Symonds, P. S., "Dynamic Load Characteristics in Plastic Bending of Beams," Jour. App. Mech., Vol. 20, No. 4, 1953, pp. 475-481.
15. Hopkins, H. G., "On the Plastic Theory of Plates," Proc. Roy. Soc. A, Vol. 241, pp. 153-179, 1957.
16. Shapiro, G. S., "On a Rigid-Plastic Annular Plate Under Impulsive Load," Journal of Applied Maths. & Mechanics (Prikl. Mat. i Mek), Vol. 23, pp. 234-241, 1959.
17. Florence, A. L., "Annular Plate Under a Transverse Line Impulse," AIAA Journal, Vol. 3, No. 9, Sept. 1965, pp. 1726-1732, 1966.
18. Witmer, E. A., Balmer, H. A., Leech, J. W., and Pian, T. H. H., "Large Dynamic Deformations of Beams, Circular Rings, Circular Plates, and Shells," AIAA Journal, Vol. 1, 1963, pp. 1848-1857.
19. Boyd, D. E., "Dynamic Deformations of Circular Membranes," Jnl. Eng. Mech. Div., Proc. ASCE, Vol. 92, No. EM3, June 1966, pp. 1-16.
20. Munday, G., and Newitt, D. M., "The Deformation of Transversely Loaded Disks Under Dynamic Loads," Phil. Trans. Roy. Soc. of London, Series A, Math. & Phys. Sciences, No. 1065, Vol. 256, pp. 1-30, Dec. 1963.



21. Johnson, W., Poynton, A., Singh, H., and Travis, F. W., "Experiments in the Underwater Explosive Stretch Forming of Clamped Circular Blanks," Int. J. Mech. Sci., Vol. 8, pp. 237-270, 1966.
22. Martin, J. B., and Symonds, P. S., "Mode Approximations for Impulsively-Loaded Rigid-Plastic Structures," Proc. ASCE, Vol. 92, No. EM5, pp. 43-66, Oct. 1966.
23. Cooper, R. M., and Shifrin, G. A., "An Experiment on Circular Plates in the Plastic Range," Proc. 2nd U. S. Nat. Cong. of App. Mechs., pp. 527-534, 1954.
24. Haythornthwaite, R. M., and Onat, E. T., "The Load Carrying Capacity of Initially Flat Circular Steel Plates Under Reversed Loading," Journal of Aero. Sci., Vol. 22, No. 12, pp. 867-869, 1955.
25. Hopkins, H. G., and Prager, W., "The Load Carrying Capacities of Circular Plates," Journal of Mechs. and Physics of Solids, Vol. 2, No. 1, pp. 1-13, 1953.
26. Onat, E. T., and Haythorthwaite, R. M., "The Load Carrying Capacity of Circular Plates at Large Deflection," Jour. Applied Mechs., Vol. 23, pp. 49-55, 1956.
27. Symonds, P. S., and Mentel, T. J., "Impulsive Loading of Plastic Beams with Axial Constraints," Jour. Mech. Phys. Solids, Vol. 6, 1958, pp. 186-202.
28. Reissner, E., "On Finite Deflections of Circular Plates," Proc. Symposium App. Math., Vol. 1, pp. 213-219, published by American Math. Soc., New York, 1949.
29. Onat, E. T., and Prager, W., "Limit Analysis of Shells of Revolution," Parts I and II, Proc. Royal Netherlands Acad. of Sci., Vol. B57, 1954, pp. 534-541, 542-548.
30. Hodge, P. G., "Yield Conditions for Rotationally Symmetric Shells Under Axisymmetric Loading," Journal of Applied Mechanics, Trans. ASME, June 1960, pp. 323-331.
31. Parkes, E. W., "The Permanent Deformation of a Cantilever Struck Transversely at its Tip," Proc. Roy. Soc. London, Vol. 228, Ser. A, 1955, pp. 462-476.
32. Bowman, F., "Introduction to Bessel Functions," Dover Publications Inc., 1958.
33. Watson, G. N., "A Treatise on the Theory of Bessel Function," 2nd Ed., Camb. Univ. Press, 1958.

34. Handbook of Mathematical Functions, U. S. Dept. of Commerce, Nat. Bureau of Standards, Ed. Abramowitz and Stegun.
35. Schmidt, P. W., J. Math. Phys. 34, pp. 169-172, 1955.

mlw

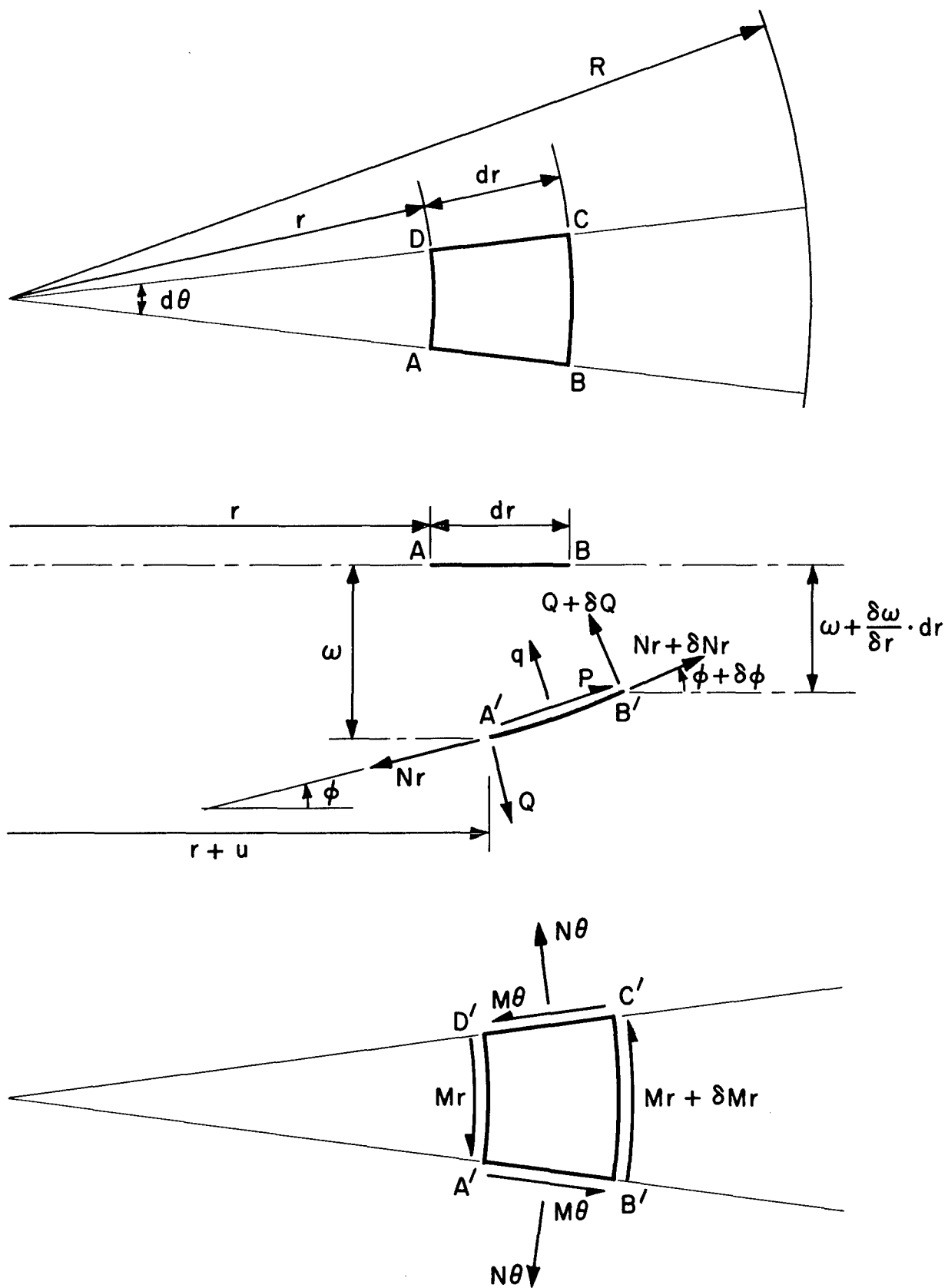


FIG. I

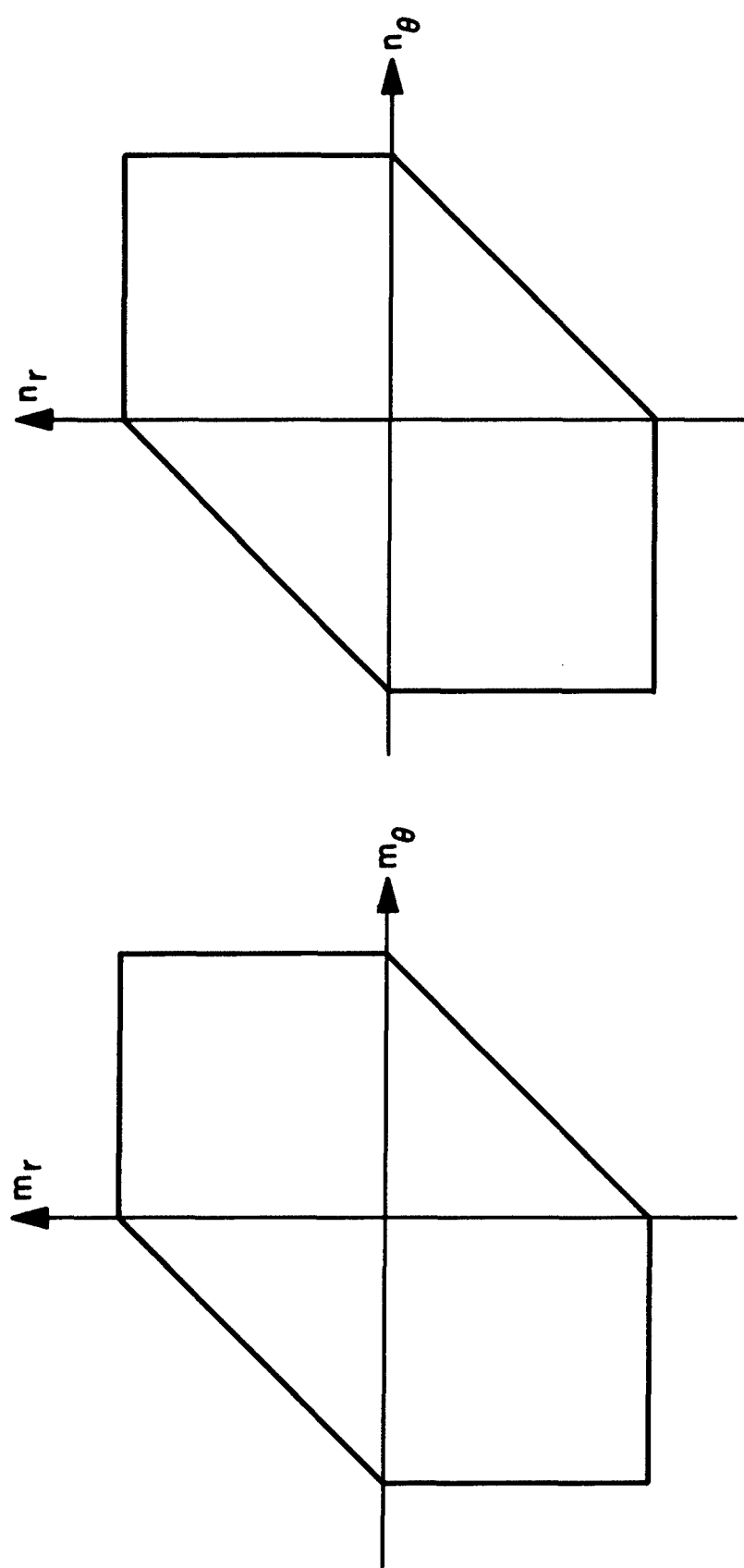


FIG. 2 YIELD CONDITION AFTER HODGE (30)

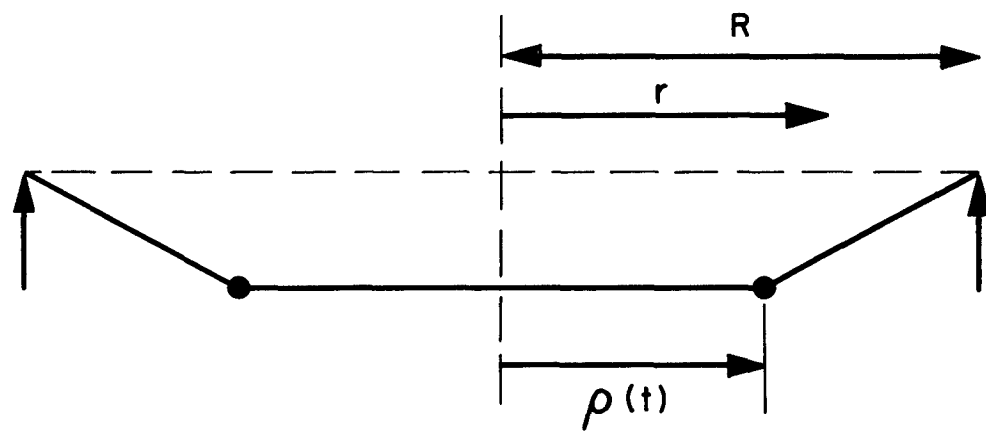
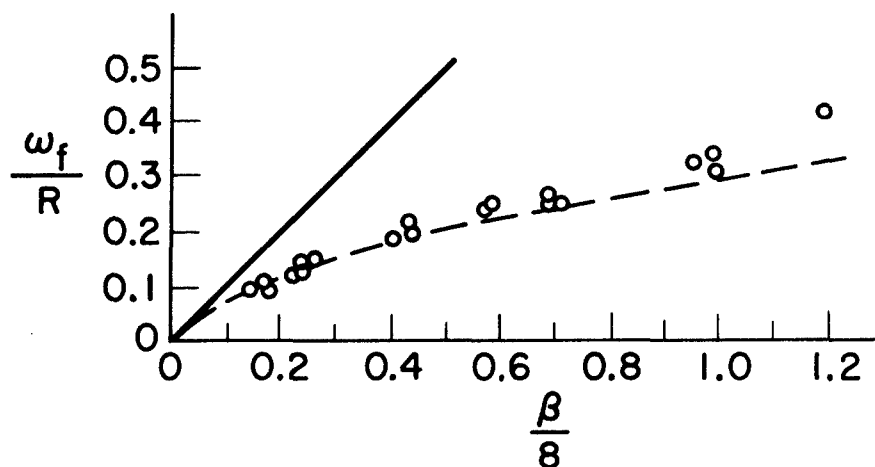


FIG.3 SIDE VIEW OF THE PLATE AT TIME  $t$ .



◦◦ EXPERIMENTAL RESULTS FOR AL 6061-T6  
R/H = 16 (AFTER FLORENCE (II))

—— BENDING ONLY THEORY (WANG (10))

--- MEMBRANE - BENDING THEORY

FIG. 4 COMPARISON OF THEORY WITH  
EXPERIMENTAL RESULTS FOR AN  
IMPULSIVELY LOADED SIMPLY  
SUPPORTED PLATE

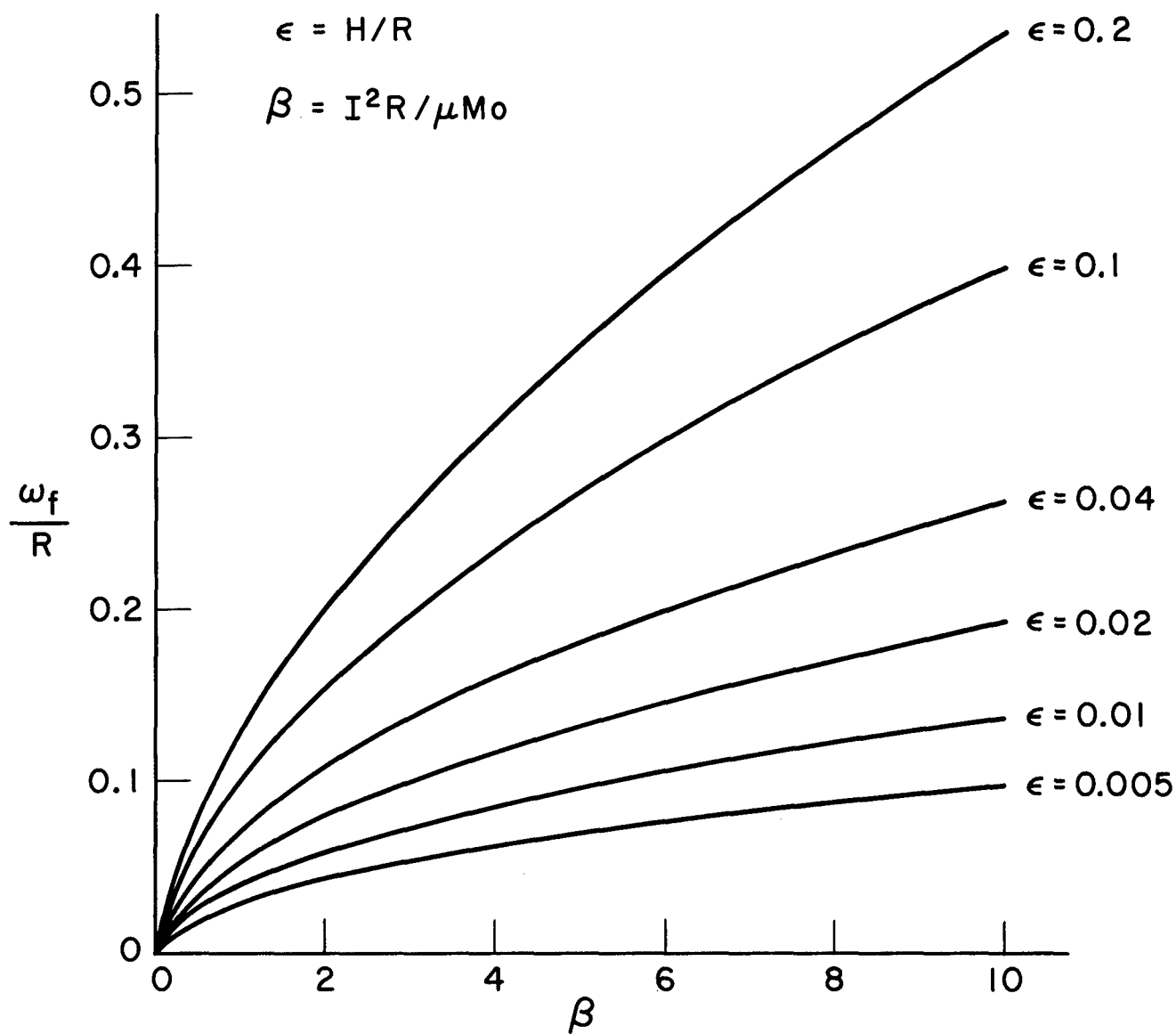


FIG. 5 DEFLECTION - IMPULSE RELATION FOR SIMPLY SUPPORTED CIRCULAR PLATES